The Motion of a Solid in a Fluid.

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IN the following paper I have given a brief account of some of the most important work that has been done upon this problem, together with some additions to the theory which I believe to be new. The method that I have given of transformation by means of elliptic coordinates seems to me to be very simple and practical, depending, as it does, upon the most elementary properties of the coefficients in a certain system of linear equations.

The fluid under consideration is assumed to be perfect, incompressible and extending to infinity in all directions; and further, the space occupied by the fluid is supposed simply-connected and consequently the velocity potential single-valued. A velocity potential will exist; as we assume that the fluid is originally at rest, and that the entire motion of the system is due to the motion of the solid, and in consequence there will be no rotational motion generated among the fluid particles.

Designate by u, v, w the component velocities of translation of a point in the body with respect to a set of rectangular axes x, y, z fixed in the body, and by p, q, r the component angular velocities of the body around these axes. Now, letting n denote the outer normal to the surface of the body, we have for the determination of the velocity potential ϕ the equation

1. $$\frac{\partial \varphi}{\partial n} = (u + zq - yr)\cos(n, x) + (v + xr - zp)\cos(n, y) + (w + yp - xq)\cos(n, z).$$

Since the fluid is to be at rest at infinity, the first derivatives of ϕ with respect to x, y and z will vanish for infinitely great values of these variables; and since $\Delta^2\phi = 0$ throughout the entire space, and ϕ with its derivatives is single-valued and continuous, we can write

2. $$\phi = u\phi_1 + v\phi_2 + w\phi_3 + p\phi_4 + q\phi_5 + r\phi_6,$$

a linear equation in the six quantities u, v, &c. The six functions ϕ_1, ϕ_2, &c., satisfy the equation $\Delta^2\phi = 0$, and, at the surface of the body, the relations

3. $$\begin{aligned} \frac{\partial \varphi_1}{\partial n} &= \cos(n, x), & \frac{\partial \varphi_4}{\partial n} &= y\cos(n, z) - z\cos(n, y), \\ \frac{\partial \varphi_2}{\partial n} &= \cos(n, y), & \frac{\partial \varphi_5}{\partial n} &= z\cos(n, x) - x\cos(n, z), \\ \frac{\partial \varphi_3}{\partial n} &= \cos(n, z), & \frac{\partial \varphi_6}{\partial n} &= x\cos(n, y) - y\cos(n, x), \end{aligned}$$

The entire motion being due to the motion of the solid, we know that the energy of the system will be a quadratic function of the six quantities u, v, w, p, q, r. Denote the energy by T and we have,

$$\begin{aligned} 2T = {} & a_{11}u^2 + a_{22}v^2 + a_{33}w^2 + 2a_{12}uv + 2a_{13}uw + 2a_{23}vw \\ & + a_{44}p^2 + a_{55}q^2 + a_{66}r^2 + 2a_{45}pq + 2a_{46}pr + 2a_{56}qr \\ & + 2p\,[a_{14}u + a_{24}v + a_{34}w] \\ & + 2q\,[a_{15}u + a_{25}v + a_{35}w] \\ & + 2r\,[a_{16}u + a_{26}v + a_{36}w]\,, \end{aligned} \tag{4}$$

the coefficients a_{ij} being constants depending upon the shape of the body and the distribution of mass in its interior. If we divide the energy T into two parts, T' and T'', the former may denote that portion of the entire energy due to the fluid—the latter, that to the body; then the coefficients a_{ij} are also divided into two parts and we may write $a_{ij} = a'_{ij} + a''_{ij}$. Kirchhoff has shown that

$$a'_{ij} = -\rho \int ds \phi_i \frac{\partial \varphi_j}{\partial n} = -\rho \int ds \phi_j \frac{\partial \varphi_i}{\partial n}.$$

Of course i and j have values from 1 to 6 inclusive. The values of the coefficients a''_{ij} are easily obtained from the expression for the energy of the solid, or

$$\begin{aligned} 2T'' = \int dm \{ & (u^2 + v^2 + w^2) + (y^2 + z^2)\,p^2 + (x^2 + z^2)\,q^2 + (x^2 + y^2)\,r^2 \\ & + 2x\,(vr - wq) + 2y\,(wp - ur) + 2z\,(uq - vp) \\ & - 2yzqr - 2zxrp - 2xypq \}\,. \end{aligned}$$

We will now take up the Kirchhoffian equations of the motion of the solid, and for brevity write

$$\begin{aligned} U &= \frac{\partial T}{\partial u}\,, & P &= \frac{\partial T}{\partial p}\,, \\ V &= \frac{\partial T}{\partial v}\,, & Q &= \frac{\partial T}{\partial q}\,, \\ W &= \frac{\partial T}{\partial w}\,, & R &= \frac{\partial T}{\partial r}\,. \end{aligned} \tag{5}$$

These equations are then

$$\begin{aligned} \frac{dU}{dt} &= rV - qW\,, & \frac{dP}{dt} &= wV - vW + rQ - qR\,, \\ \frac{dV}{dt} &= pW - rU\,, & \frac{dQ}{dt} &= uW - wU + pR - rP\,, \\ \frac{dW}{dt} &= qU - pV\,, & \frac{dR}{dt} &= vU - uV + qP - pQ\,. \end{aligned} \tag{6}$$

Concerning the forces that act upon the body, we know that, whatever be the motion at any instant, we can conceive it generated instantaneously from

rest by a properly chosen impulse applied to the body; this impulse, according to the method of Poinsot, consisting of a force and a couple, whose axis is in the direction of the force. The quantities U, V, &c., are then the components with respect to the axes x, y, z of this force and couple; and the above equations show that these quantities vary only with the motion of the axes to which they are referred. Kirchhoff has observed that a particular solution of the above equations is obtained by supposing u, v, w constant and $p = q = r = 0$, provided we have

7. $$\frac{U}{u} = \frac{V}{v} = \frac{W}{w},$$

or,

7′. $$\frac{a_{11}u + a_{12}v + a_{13}w}{u} = \frac{a_{21}u + a_{22}v + a_{23}w}{v} = \frac{a_{31}u + a_{32}v + a_{33}w}{w};$$

(of course $a_{ij} = a_{ji}$), that is, provided the velocity, of which u, v, w are the rectangular components, be parallel to one of the principal axes of the ellipsoid,

8. $$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz = \text{const.}$$

Calling λ the common value of the above quantities, we have for the determination of λ the cubic,

9. $$\begin{vmatrix} a_{11} - \lambda, & a_{12}, & a_{13}, \\ a_{21}, & a_{22} - \lambda, & a_{23}, \\ a_{31}, & a_{32}, & a_{33} - \lambda, \end{vmatrix} = 0.$$

The eliminant of the equations of motion is

$$\begin{vmatrix} 0, & 0, & 0, & 0, & -W, & V, \\ 0, & 0, & 0, & W, & 0, & -U, \\ 0, & 0, & 0, & -V, & U, & 0, \\ 0, & -W, & V, & 0, & -R, & Q, \\ W, & 0, & -U, & R, & 0, & -P, \\ -V, & U, & 0, & -Q, & P, & 0, \end{vmatrix},$$

which is obviously equal to zero, showing that there are only five independent relations to be satisfied in our equations, viz: the ratios $u : v : w : p : q : r$.

Reverting now to our value for T, we obtain for U, V, &c., the following values:

10. $$\begin{aligned} U &= a_{11}u + a_{12}v + a_{13}w + a_{14}p + a_{15}q + a_{16}r, \\ V &= a_{21}u + a_{22}v + a_{23}w + a_{24}p + a_{25}q + a_{26}r, \\ W &= a_{31}u + a_{32}v + a_{33}w + a_{34}p + a_{35}q + a_{36}r, \\ P &= a_{41}u + a_{42}v + a_{43}w + a_{44}p + a_{45}q + a_{46}r, \\ Q &= a_{51}u + a_{52}v + a_{53}w + a_{54}p + a_{55}q + a_{56}r, \\ R &= a_{61}u + a_{62}v + a_{63}w + a_{64}p + a_{65}q + a_{66}r. \end{aligned}$$

Denote by ∇ the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{vmatrix}$$

According to the well-known conditions that must be satisfied in order that T may be positive, we must have

$$\nabla, \; \frac{\partial\nabla}{\partial a_{11}}, \; \frac{\partial^2\nabla}{\partial a_{11}\partial a_{22}} \cdots \cdot \frac{\partial^6\nabla}{\partial a_{11}\ldots\partial a_{66}},$$

all positive and different from zero, a point of great importance.

Now from the last equations we can determine the velocities in terms of the forces, and for this determination we have

$$\begin{aligned} u &= \nabla^{-1}\left[\frac{\partial\nabla}{\partial a_{11}}U + \frac{\partial\nabla}{\partial a_{12}}V + \frac{\partial\nabla}{\partial a_{13}}W + \frac{\partial\nabla}{\partial a_{14}}P + \frac{\partial\nabla}{\partial a_{15}}Q + \frac{\partial\nabla}{\partial a_{16}}R\right], \\ v &= \nabla^{-1}\left[\frac{\partial\nabla}{\partial a_{21}}U + \frac{\partial\nabla}{\partial a_{22}}V + \frac{\partial\nabla}{\partial a_{23}}W + \frac{\partial\nabla}{\partial a_{24}}P + \frac{\partial\nabla}{\partial a_{25}}Q + \frac{\partial\nabla}{\partial a_{26}}R\right], \\ w &= \nabla^{-1}\left[\frac{\partial\nabla}{\partial a_{31}}U + \frac{\partial\nabla}{\partial a_{32}}V + \frac{\partial\nabla}{\partial a_{33}}W + \frac{\partial\nabla}{\partial a_{34}}P + \frac{\partial\nabla}{\partial a_{35}}Q + \frac{\partial\nabla}{\partial a_{36}}R\right], \\ p &= \nabla^{-1}\left[\frac{\partial\nabla}{\partial a_{41}}U + \frac{\partial\nabla}{\partial a_{42}}V + \frac{\partial\nabla}{\partial a_{43}}W + \frac{\partial\nabla}{\partial a_{44}}P + \frac{\partial\nabla}{\partial a_{45}}Q + \frac{\partial\nabla}{\partial a_{46}}R\right], \\ q &= \nabla^{-1}\left[\frac{\partial\nabla}{\partial a_{51}}U + \frac{\partial\nabla}{\partial a_{52}}V + \frac{\partial\nabla}{\partial a_{53}}W + \frac{\partial\nabla}{\partial a_{54}}P + \frac{\partial\nabla}{\partial a_{55}}Q + \frac{\partial\nabla}{\partial a_{56}}R\right], \\ r &= \nabla^{-1}\left[\frac{\partial\nabla}{\partial a_{61}}U + \frac{\partial\nabla}{\partial a_{62}}V + \frac{\partial\nabla}{\partial a_{63}}W + \frac{\partial\nabla}{\partial a_{64}}P + \frac{\partial\nabla}{\partial a_{65}}Q + \frac{\partial\nabla}{\partial a_{66}}R\right]. \end{aligned} \tag{11}$$

For equations 6, it is easy to see that we have the three integrals

$$\begin{aligned} 2T &= L, \\ U^2 + V^2 + W^2 &= M, \\ UP + VQ + WR &= N, \end{aligned} \tag{12}$$

L, M and N denoting arbitrary constants. These are the general integrals given by Kirchhoff. If we introduce a set of axes ξ, η, ζ fixed in the fluid, we of course have, α, β, γ, $\alpha_1 \ldots \gamma_3$ being functions of the time,

$$\begin{aligned} \xi &= \alpha + \alpha_1 x + \beta_1 y + \gamma_1 z, \\ \eta &= \beta + \alpha_2 x + \beta_2 y + \gamma_2 z, \\ \zeta &= \gamma + \alpha_3 x + \beta_3 y + \gamma_3 z, \end{aligned} \tag{13}$$

the quantities u, v, w, p, q, r being connected with the quantities α, β, . . . by the known relations

14.
$$u = \alpha_1 \frac{d\alpha}{dt} + \beta_1 \frac{d\beta}{dt} + \gamma_1 \frac{d\gamma}{dt}, \quad \&c., \quad \&c.$$
$$p = \alpha_2 \frac{d\alpha_3}{dt} + \beta_2 \frac{d\beta_3}{dt} + \gamma_2 \frac{d\gamma_3}{dt}, \quad \&c., \quad \&c.$$

and also

$$\frac{d\alpha_1}{dt} = \alpha_3 q - \alpha_2 r, \quad \frac{d\beta_1}{dt} = \beta_3 q - \beta_2 r, \quad \frac{d\gamma_1}{dt} = \gamma_3 q - \gamma_2 r.$$

. . . .

These last nine equations are of the form (as remarked by Clebsch),

$$\frac{dA_1}{dt} = A_3 q - A_2 r,$$
$$\frac{dA_2}{dt} = A_1 r - A_3 p,$$
$$\frac{dA_3}{dt} = A_2 p - A_1 q;$$

from which we have immediately the integral

$$A_1^2 + A_2^2 + A_3^2 = \text{const.},$$

the const. being for our case $= 1$. Multiply now equations 6 by A_1, A_2, A_3 respectively, and add, observing the above relations, and we have

$$\frac{d}{dt}(A_1 U + A_2 V + A_3 W) = 0,$$

which gives us the three Kirchhoffian integrals

15.
$$\alpha_1 U + \alpha_2 V + \alpha_3 W = L',$$
$$\beta_1 U + \beta_2 V + \beta_3 W = M',$$
$$\gamma_1 U + \gamma_2 V + \gamma_3 W = N'.$$

To these integrals for determining the position of the body we can add three more, viz:

16.
$$\alpha_1 P + \alpha_2 Q + \alpha_3 R = l + \beta N' - \gamma M',$$
$$\beta_1 P + \beta_2 Q + \beta_3 R = m + \gamma L' - \alpha N',$$
$$\gamma_1 P + \gamma_2 Q + \gamma_3 R = n + \alpha M' - \beta L',$$

the following relations obviously connecting the constants:

17.
$$L'^2 + M'^2 + N'^2 = M,$$
$$L'l + M'm + N'n = N.$$

The particular case where the body has its mass distributed symmetrically with respect to three mutually perpendicular planes, *i. e.* where T takes the form

$$a_{11}u^2 + a_{22}v^2 + a_{33}w^2 + a_{44}p^2 + a_{55}q^2 + a_{66}r^2,$$

has been discussed in a very elegant manner by Weber in a recent number of the Mathematische Annalen. The investigation is made in the first place to depend upon a remarkable property of the ϑ-functions of two variables, and secondly, the author proceeds to the direct integration of the differential equations by means of hyperelliptic integrals.

For the particular case where the body is an ellipsoid, the values of a_{ij}, determined by the formulæ

$$a_{ij} = -\rho \int ds \phi_i \frac{\partial \varphi_j}{\partial n}$$

can be readily seen to depend upon elliptic functions, the integration extending over the entire surface of the ellipsoid of which ds is an element. The case where the body possesses a surface of revolution and a symmetrical distribution of mass, has been very fully discussed by Kirchhoff (in Vol. 71 of Crelle's Journal) who makes the solution of the problem depend upon elliptic functions. The form of T for this case is given by

$$2T = a_{11}u^2 + a_{22}(v^2 + w^2) + a_{44}p^2 + a_{55}(q^2 + r^2),$$

the constants reducing to only four. In Vol. 12 of the Mathematische Annalen, Köpcke has discussed the same problem by aid of the ϑ-functions, obtaining results which are very convenient for numerical computation.

Steady Motion.

The method employed in the following brief examination of the *steady* motion of a solid in a fluid was suggested to me by reading Routh's Essay "On the Stability of a Given State of Motion." I believe the results stated to be new, though I would not venture to make any positive assertion to that effect. I can simply say that the investigation is original and the results obtained seem of interest. "A steady motion is such that the same change of motion follows from the same initial disturbance at whatever instant the disturbance is communicated to the system." The conditions for steady motion of the solid are given by the relations

α. $$\frac{dU}{dt} = \frac{dV}{dt} = \frac{dW}{dt} = 0,$$

β. $$\frac{dP}{dt} = \frac{dQ}{dt} = \frac{dR}{dt} = 0.$$

The conditions α are satisfied by making

18. $$\frac{U}{p} = \frac{V}{q} = \frac{W}{r} = \lambda,$$

or,

18'. $$\frac{a_{11}u + a_{12}v + a_{13}w}{p} = \frac{a_{21}u + a_{22}v + a_{23}w}{q} = \frac{a_{31}u + a_{32}v + a_{33}w}{r} = \lambda .$$

The equations β give now

19. $$\frac{P - \lambda u}{p} = \frac{Q - \lambda v}{q} = \frac{R - \lambda w}{r} = \mu .$$

These last equations can evidently be replaced by

20. $$\frac{UP + VQ + WR - \lambda(uU + vV + wW)}{pU + qV + rW} = \mu .$$

or,

$$(\lambda u + \mu p)\, U + (\lambda v + \mu q)\, V + (\lambda w + \mu r)\, W = \text{const.},$$

which is identical with the known relation

$$UP + VQ + WR = \text{const.}$$

From the expressions for λ and μ we see that it is possible for the body to have an infinite number of steady motions, without making any restrictions as to the form of the body or the distribution of mass in its interior. These steady motions being each produced by a certain *impulse*, will consist in general of a translation in the direction of, and a rotation round, the axis of the component couple. The locus of the system of axes is a ruled surface, whose position with respect to the body is of course invariable. The component velocities of the body with reference to the axes x, y, z fixed in the body are

$$u + zq - yr ,$$
$$v + xr - zp ,$$
$$w + yp - xq ,$$

or, as these may be written

$$\frac{1}{\lambda}\,[P - \mu p + \lambda\,(zq - yr)] ,$$
$$\frac{1}{\lambda}\,[Q - \mu q + \lambda\,(xr - zp)] ,$$
$$\frac{1}{\lambda}\,[R - \mu r + \lambda\,(yp - xq)] .$$

Substituting for U, V, &c., their values in the equations giving λ and μ, we obtain the system of linear equations

21. $$\begin{aligned} &a_{11}u + a_{12}v + a_{13}w + (a_{14} - \mu)\, p + a_{15}q + a_{16}r = 0 , \\ &a_{21}u + a_{22}v + a_{23}w + a_{24}p + (a_{25} - \lambda)\, q + a_{26}r = 0 , \\ &a_{31}u + a_{32}v + a_{33}w + a_{34}p + a_{35}q + (a_{36} - \lambda)\, r = 0 , \\ &(a_{41} - \lambda)\, u + a_{42}v + a_{43}w + (a_{44} - \mu)\, p + a_{45}q + a_{46}r = 0 , \\ &a_{51}u + (a_{52} - \lambda)\, v + r_{53}w + a_{54}p + (a_{55} - \mu)\, q + a_{56}r = 0 , \\ &a_{61}u + a_{62}v + (a_{63} - \lambda)\, w + a_{64}p + a_{65}q + (a_{66} - \mu)\, r = 0 . \end{aligned}$$

Eliminating the ratios $u:v:$&c., we have for the relation connecting the quantities λ and μ

22. $$\nabla_{\lambda\mu} \equiv \begin{vmatrix} a_{11}, & a_{12}, & a_{13}, & a_{14}-\lambda, & a_{15}, & a_{16} \\ a_{21}, & a_{22}, & a_{23}, & a_{24}, & a_{25}-\lambda, & a_{26} \\ a_{31}, & a_{32}, & a_{33}, & a_{34}, & a_{35}, & a_{36}-\lambda \\ a_{41}-\lambda, & a_{42}, & a_{43}, & a_{44}-\mu, & a_{45}, & a_{46} \\ a_{51}, & a_{52}-\lambda, & a_{53}, & a_{54}, & a_{55}-\mu, & a_{56} \\ a_{61}, & a_{62}, & a_{63}-\lambda, & a_{64}, & a_{65}, & a_{66}-\mu \end{vmatrix} = 0.$$

This equation affords us the means of determining either λ or μ, provided we assume a determinate value for one of these quantities. Assume for λ some arbitrary value, then the equation $\nabla_{\lambda\mu} = 0$ is of the third degree in μ, and we have thus, for any one value of λ, three corresponding values of μ. We have now, for the complete specification of the motion,

23. $$u:v:w:p:q:r =$$
$$\frac{\partial \nabla_{\lambda\mu}}{\partial a_{11}} : \frac{\partial \nabla_{\lambda\mu}}{\partial a_{12}} : \frac{\partial \nabla_{\lambda\mu}}{\partial a_{13}} : \frac{\partial \nabla_{\lambda\mu}}{\partial a_{14}} : \frac{\partial \nabla_{\lambda\mu}}{\partial a_{15}} : \frac{\partial \nabla_{\lambda\mu}}{\partial a_{16}},$$

substituting in the minors $\frac{\partial \nabla_{\lambda\mu}}{\partial a_{16}}$ the assumed value of λ and the determined value of μ.

If we give μ a determinate value, we have $\nabla_{\lambda\mu} = 0$, an equation of the sixth degree for finding the corresponding values of λ. If we make $\mu = 0$, it is known that the roots of $\nabla_{\lambda\mu} = 0$ will all be real, three positive and three negative, and the motion in this case will be completely determined.

For simplicity we may assume the axes of x, y, z parallel to the three directions of permanent translation, which is equivalent to making

$$a_{12} = a_{13} = a_{23} = 0;$$

we have then from the first three of equations 16,

24. $$u = \frac{(a_{14}-\lambda)p + a_{15}q + a_{16}r}{a_{11}},$$
$$v = \frac{a_{24}p + (a_{25}-\lambda)q + a_{26}r}{a_{22}},$$
$$w = \frac{a_{34}p + a_{35}q + (a_{36}-\lambda)r}{a_{33}},$$

Substituting these values in the last three of equations 16, we have

$$\left[\frac{(a_{14}-\lambda)^2}{a_{11}}+\frac{a_{24}^2}{a_{22}}+\frac{a_{34}^2}{a_{33}}+a_{44}\right]p+\left[\frac{(a_{14}-\lambda)\,a_{15}}{a_{11}}+\frac{a_{24}(a_{25}-\lambda)}{a_{22}}+\frac{a_{34}a_{35}}{a_{33}}+a_{45}\right]q$$
$$+\left[\frac{(a_{14}-\lambda)\,a_{16}}{a_{11}}+\frac{a_{24}a_{26}}{a_{22}}+\frac{a_{34}(a_{36}-\lambda)}{a_{33}}+a_{46}\right]r=\mu p,$$
$$\left[\frac{(a_{14}-\lambda)\,a_{15}}{a_{11}}+\frac{a_{24}(a_{25}-\lambda)}{a_{22}}+\frac{a_{34}a_{35}}{a_{33}}+a_{45}\right]p+\left[\frac{a_{15}^2}{a_{11}}+\frac{(a_{25}-\lambda)^2}{a_{22}}+\frac{a_{35}^2}{a_{33}}+a_{55}\right]q$$
$$+\left[\frac{a_{15}a_{16}}{a_{11}}+\frac{(a_{25}-\lambda)\,a_{26}}{a_{22}}+\frac{a_{35}(a_{36}-\lambda)}{a_{33}}+a_{56}\right]r=\mu q,$$
$$\left[\frac{(a_{14}-\lambda)\,a_{16}}{a_{11}}+\frac{a_{24}a_{26}}{a_{22}}+\frac{a_{34}(a_{36}-\lambda)}{a_{33}}+a_{46}\right]p+\left[\frac{a_{15}a_{16}}{a_{11}}+\frac{(a_{25}-\lambda)\,a_{26}}{a_{22}}+\frac{a_{35}(a_{36}-\lambda)}{a_{33}}+a_{56}\right]q$$
$$+\left[\frac{a_{16}^2}{a_{11}}+\frac{a_{26}^2}{a_{22}}+\frac{(a_{36}-\lambda)^2}{a_{33}}+a_{66}\right]r=\mu r.$$

These may obviously be written briefly in the form

25.
$$\begin{aligned} Ep+G'q+F'r&=\mu p\,,\\ G'p+Fq+E'r&=\mu q\,,\\ F'p+E'q+Gr&=\mu r\,. \end{aligned}$$

We have then for μ the cubic

26.
$$\begin{vmatrix} E-\mu, & G', & F' \\ G', & F-\mu, & E' \\ F', & E', & G-\mu \end{vmatrix}=0.$$

Thus the directions of the three axes corresponding to an assumed value of λ are at right angles to each other, but need not intersect; and, in general, no two values of μ will coincide with each other. Taking then a series of values of λ. and finding the corresponding values of μ, we will have, as the locus of the axes, a ruled surface of three distinct nappes.

Assume, in equations 6, that the only force which acts upon the body is the couple whose components are P, Q and R, *i. e.* make

$$U=V=W=0;$$

the equations are then satisfied by writing

27.
$$\frac{P}{p}=\frac{Q}{q}=\frac{R}{r}=\mu\,,$$

each of these ratios being the value of μ for $\lambda=0$. Calling μ_1, μ_2, μ_3 the three corresponding values of μ, write

28.
$$\begin{aligned} P&=\mu p\,,\\ Q&=\mu q\,,\\ R&=\mu r\,. \end{aligned}$$

These, substituted in equations 6, give us

$$\mu_1 \frac{dp}{dt} = (\mu_2 - \mu_3)\, qr\,,$$

29. $$\mu_2 \frac{dq}{dt} = (\mu_3 - \mu_1)\, rp\,,$$

$$\mu_3 \frac{dr}{dt} = (\mu_1 - \mu_2)\, pq\,;$$

these three equations are identical in form with Euler's equations for the rotation of a rigid body, and their solution can, consequently, be regarded as known. The coefficients μ_1, μ_2, μ_3 depend, in our case, not only upon the body but upon the density of the fluid;—if this latter was supposed equal to zero, the problem would coincide with that of the rotation of a free rigid body.

Assume that the body possesses three planes of symmetry, or, to fix the idea, assume the body to be an ellipsoid. The expression for the energy becomes in this case

$$2T = a_{11}u^2 + a_{22}v^2 + a_{33}w^2 + a_{44}p^2 + a_{55}q^2 + a_{66}r^2\,,$$

the other terms all vanishing since the sign of T must remain unchanged if we change u into $-u$, v into $-v$, &c. We have now

$$U = a_{11}u\,, \quad P = a_{44}p\,,$$

30. $$V = a_{22}v\,, \quad Q = a_{55}q\,,$$

$$W = a_{33}w\,, \quad R = a_{66}r\,,$$

giving, of course, for steady motion,

$$\frac{du}{dt} = \frac{dv}{dt} = \frac{dw}{dt} = 0\,,$$

$$\frac{dp}{dt} = \frac{dq}{dt} = \frac{dr}{dt} = 0\,,$$

or the axes of steady motion are the axes of the ellipsoid.

Motion of the Fluid.

The equations giving the motions of the fluid particles relatively to the body are

$$\frac{dx}{dt} = \frac{\partial\varphi}{\partial x} - u - zq + yr\,,$$

31. $$\frac{dy}{dt} = \frac{\partial\varphi}{\partial y} - v - xr + zp\,,$$

$$\frac{dz}{dt} = \frac{\partial\varphi}{\partial z} - w - yp + xq\,,$$

φ denoting the velocity potential.

In the investigation of the motion of the fluid particles due to the motion of a body of given form, it is often desirable to transform the equations of motion from rectangular to curvilinear coordinates, and to this transformation we will now turn our attention. Taking the quantities $\lambda_1, \lambda_2, \lambda_3$ as the variable parameters in a certain system of curvilinear coordinates, let us suppose that we have the relations

32. $$\lambda_1 = F_1(x, y, z), \quad \lambda_2 = F_2(x, y, z), \quad \lambda_3 = F_3(x, y, z),$$ and 33. $$x = f_1(\lambda_1, \lambda_2, \lambda_3), \quad y = f_2(\lambda_1, \lambda_2, \lambda_3), \quad z = f_3(\lambda_1, \lambda_2, \lambda_3).$$

The known conditions that the surfaces $\lambda_1 = \text{const.}$, $\lambda_2 = \text{const.}$, $\lambda_3 = \text{const.}$, should be orthogonal are

34. $$\frac{\partial\lambda_1}{\partial x}\frac{\partial\lambda_2}{\partial x} + \frac{\partial\lambda_1}{\partial y}\frac{\partial\lambda_2}{\partial y} + \frac{\partial\lambda_1}{\partial z}\frac{\partial\lambda_2}{\partial z} = 0, \text{ \&c.,}$$

and

35. $$\frac{\partial x}{\partial\lambda_1}\frac{\partial x}{\partial\lambda_2} + \frac{\partial y}{\partial\lambda_1}\frac{\partial y}{\partial\lambda_2} + \frac{\partial z}{\partial\lambda_1}\frac{\partial z}{\partial\lambda_2} = 0, \text{ \&c.,}$$

We have further,

36. $$\begin{aligned} dx &= \frac{\partial f_1}{\partial\lambda_1} d\lambda_1 + \frac{\partial f_1}{\partial\lambda_2} d\lambda_2 + \frac{\partial f_1}{\partial\lambda_3} d\lambda_3, \\ dy &= \frac{\partial f_2}{\partial\lambda_1} d\lambda_1 + \frac{\partial f_2}{\partial\lambda_2} d\lambda_2 + \frac{\partial f_2}{\partial\lambda_3} d\lambda_3, \\ dz &= \frac{\partial f_3}{\partial\lambda_1} d\lambda_1 + \frac{\partial f_3}{\partial\lambda_2} d\lambda_2 + \frac{\partial f_3}{\partial\lambda_3} d\lambda_3. \end{aligned}$$

These give, as is well known,

$$\begin{aligned} E^2 d\lambda_1 &= \frac{\partial x}{\partial\lambda_1} dx + \frac{\partial y}{\partial\lambda_1} dy + \frac{\partial z}{\partial\lambda_1} dz, \\ F^2 d\lambda_2 &= \frac{\partial x}{\partial\lambda_2} dx + \frac{\partial y}{\partial\lambda_2} dy + \frac{\partial z}{\partial\lambda_2} dz, \\ G^2 d\lambda_3 &= \frac{\partial x}{\partial\lambda_3} dx + \frac{\partial y}{\partial\lambda_3} dy + \frac{\partial z}{\partial\lambda_3} dz, \end{aligned}$$

whence

37. $$\begin{aligned} E^2 &= \left(\frac{\partial x}{\partial\lambda_1}\right)^2 + \left(\frac{\partial y}{\partial\lambda_1}\right)^2 + \left(\frac{\partial z}{\partial\lambda_1}\right)^2, \\ F^2 &= \left(\frac{\partial x}{\partial\lambda_2}\right)^2 + \left(\frac{\partial y}{\partial\lambda_2}\right)^2 + \left(\frac{\partial z}{\partial\lambda_2}\right)^2, \\ G^2 &= \left(\frac{\partial x}{\partial\lambda_3}\right)^2 + \left(\frac{\partial y}{\partial\lambda_3}\right)^2 + \left(\frac{\partial z}{\partial\lambda_3}\right)^2, \end{aligned}$$

and

38.
$$\frac{1}{E^2}=\left(\frac{\partial\lambda_1}{\partial x}\right)^2+\left(\frac{\partial\lambda_1}{\partial y}\right)^2+\left(\frac{\partial\lambda_1}{\partial z}\right)^2,$$
$$\frac{1}{F^2}=\left(\frac{\partial\lambda_2}{\partial x}\right)^2+\left(\frac{\partial\lambda_2}{\partial y}\right)^2+\left(\frac{\partial\lambda_2}{\partial z}\right)^2,$$
$$\frac{1}{G^2}=\left(\frac{\partial\lambda_3}{\partial x}\right)^2+\left(\frac{\partial\lambda_3}{\partial y}\right)^2+\left(\frac{\partial\lambda_3}{\partial z}\right)^2.$$

The F entering here has, of course, no connection with the F in the above functional expressions. Our equations for the motion of a fluid particle now give, by very easy reductions,

39.
$$E^2\frac{d\lambda_1}{dt}=\frac{\partial\varphi}{\partial\lambda_1}-\left(u\frac{\partial x}{\partial\lambda_1}+v\frac{\partial y}{\partial\lambda_1}+w\frac{\partial z}{\partial\lambda_1}\right)-\left(py\frac{\partial z}{\partial\lambda_1}\right),$$
$$F^2\frac{d\lambda_2}{dt}=\frac{\partial\varphi}{\partial\lambda_2}-\left(u\frac{\partial x}{\partial\lambda_2}+v\frac{\partial y}{\partial\lambda_2}+w\frac{\partial z}{\partial\lambda_2}\right)-\left(py\frac{\partial z}{\partial\lambda_2}\right),$$
$$G^2\frac{d\lambda_3}{dt}=\frac{\partial\varphi}{\partial\lambda_3}-\left(u\frac{\partial x}{\partial\lambda_3}+v\frac{\partial y}{\partial\lambda_3}+w\frac{\partial z}{\partial\lambda_3}\right)-\left(py\frac{\partial z}{\partial\lambda_3}\right),$$

where

40.
$$\left(py\frac{\partial z}{\partial\lambda_i}\right)=\begin{vmatrix} p\,, & q\,, & r \\ x\,, & y\,, & z \\ \frac{\partial x}{\partial\lambda_i}, & \frac{\partial y}{\partial\lambda_i}, & \frac{\partial z}{\partial\lambda_i}\end{vmatrix}$$

Suppose we have now a system of orthogonal surfaces of the second order given by the equations

41.
$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}+\frac{z^2}{c^2+\lambda_1}=1,$$
$$\frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}+\frac{z^2}{c^2+\lambda_2}=1,$$
$$\frac{x^2}{a^2+\lambda_3}+\frac{y^2}{b^2+\lambda_3}+\frac{z^2}{c^2+\lambda_3}=1,$$

where, for $\lambda_1=0$ the first equation becomes the equation of the surface of the body if this is an ellipsoid.

The values of x, y and z, derivable from these equations, are, of course, well known, but the following direct transformation, which I believe to be new, is so brief, and depends upon such elementary properties of a system of linear equations, that I give it before proceeding further with the problem in hand. Consider the group of n linear equations:

42.
$$\frac{x_1}{\alpha_1}+\frac{x_2}{\alpha_2}+\frac{x_3}{\alpha_3}+\ldots\frac{x_n}{\alpha_n}=1,$$
$$\frac{x_1}{\beta_1}+\frac{x_2}{\beta_2}+\frac{x_3}{\beta_3}+\ldots\frac{x_n}{\beta_n}=1,$$
$$\cdots\cdots\cdots\cdots$$
$$\frac{x_1}{\nu_1}+\frac{x_2}{\nu_2}+\frac{x_3}{\nu_3}+\ldots\frac{x_n}{\nu_n}=1,$$

having the following relations among the quantities α, β, &c.:

$$\alpha_1 - \alpha_2 = \beta_1 - \beta_2 = \gamma_1 - \gamma_2 = \ldots \nu_1 - \nu_2\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

43. $$\alpha_1 - \alpha_i = \beta_1 - \beta_i = \gamma_1 - \gamma_i = \ldots \nu_1 - \nu_i\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\alpha_1 - \alpha_n = \beta_1 - \beta_n = \gamma_1 - \gamma_n = \ldots \nu_1 - \nu_n\,,$$

and

$$\alpha_1 - \beta_1 = \alpha_2 - \beta_2 = \alpha_3 - \beta_3 = \ldots \alpha_n - \beta_n\,,$$

&c., &c., &c.

Designate by ∇ the eliminant of our set of equations—that is,

$$\nabla = \begin{vmatrix} \alpha_1^{-1}, & \alpha_2^{-1}, & \alpha_3^{-1}, \ldots & \alpha_n^{-1} \\ \beta_1^{-1}, & \beta_2^{-1}, & \beta_3^{-1}, \ldots & \beta_n^{-1} \\ \cdot & \cdot & \cdot & \cdot \\ \nu_1^{-1}, & \nu_2^{-1}, & \nu_3^{-1}, \ldots & \nu_n^{-1} \end{vmatrix}$$

Then we have for the values of x_1, x_2, &c.,

44. $$\begin{aligned} x_1 &= \nabla^{-1}\left[\frac{\partial\nabla}{\partial\alpha_1^{-1}} + \frac{\partial\nabla}{\partial\beta_1^{-1}} + \ldots \frac{\partial\nabla}{\partial\nu_1^{-1}}\right], \\ x_2 &= \nabla^{-1}\left[\frac{\partial\nabla}{\partial\alpha_2^{-1}} + \frac{\partial\nabla}{\partial\beta_2^{-1}} + \ldots \frac{\partial\nabla}{\partial\nu_2^{-1}}\right], \\ &\cdots\cdots\cdots\cdots \\ a_i &= \nabla^{-1}\left[\frac{\partial\nabla}{\partial\alpha_i^{-1}} + \frac{\partial\nabla}{\partial\beta_i^{-1}} + \ldots \frac{\partial\nabla}{\partial\nu_i^{-1}}\right], \end{aligned}$$

&c., &c.

For the further determination of these quantities, write the determinant in the form

$$\nabla = \alpha_1^{-1}\beta_1^{-1}\gamma_1^{-1} \ldots \nu_1^{-1} \begin{vmatrix} 1, & \alpha_1\alpha_2^{-1}, & \alpha_1\alpha_3^{-1}, \ldots & \alpha_1\alpha_n^{-1} \\ 1, & \beta_1\beta_2^{-1}, & \beta_1\beta_3^{-1}, \ldots & \beta_1\beta_n^{-1} \\ \cdot & \cdot & \cdot & \cdot \\ 1, & \nu_1\nu_2^{-1}, & \nu_1\nu_3^{-1}, \ldots & \nu_1\nu_n^{-1} \end{vmatrix},$$

or again, by subtracting the first column from each of the others,

$$\nabla = \alpha_1^{-1}\beta_1^{-1}\gamma_1^{-1} \ldots \nu_1^{-1} \begin{vmatrix} 1, & \dfrac{\alpha_1 - \alpha_2}{\alpha_2}, & \dfrac{\alpha_1 - \alpha_3}{\alpha_3}, \ldots & \dfrac{\alpha_1 - \alpha_n}{\alpha_n} \\ 1, & \dfrac{\beta_1 - \beta_2}{\beta_2}, & \dfrac{\beta_1 - \beta_3}{\beta_3}, \ldots & \dfrac{\beta_2 - \beta_n}{\beta_n} \\ \cdot & \cdot & \cdot & \cdot \\ 1, & \dfrac{\nu_1 - \nu_2}{\nu_2}, & \dfrac{\nu_1 - \nu_3}{\nu_3}, \ldots & \dfrac{\nu_1 - \nu_n}{\nu_n} \end{vmatrix}$$

This becomes, by virtue of the assumed relations existing between the differences of the quantities, $\alpha, \beta \ldots \nu$,

$$\nabla = \frac{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\ldots(\alpha_1-\alpha_n)}{\alpha_1\beta_1\gamma_1\ldots\nu_1}\begin{vmatrix} 1, & \alpha_2^{-1}, & \alpha_3^{-1}, & \ldots \alpha_n^{-1} \\ 1, & \beta_2^{-1}, & \beta_3^{-1}, & \ldots \beta_n^{-1} \\ \cdot & \cdot & \cdot & \cdot \\ 1, & \nu_2^{-1}, & \nu_3^{-1}, & \ldots \nu_n^{-1} \end{vmatrix}.$$

The principal minors of this determinant, with reference to the elements in the first column, are the same as the corresponding principal minors of our original determinant; therefore, we have

45. $$\nabla = \frac{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\ldots(\alpha_1-\alpha_n)}{\alpha_1\beta_1\gamma_1\ldots\nu_1}\left[\frac{\partial\nabla}{\partial\alpha_1^{-1}} + \frac{\partial\nabla}{\partial\beta_1^{-1}} + \ldots \frac{\partial\nabla}{\partial\nu_1^{-1}}\right],$$

from which follows

46. $$x_1 = \frac{\alpha_1\beta_1\gamma_1\ldots\nu_1}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\ldots(\alpha_1-\alpha_n)},$$

and in general

$$x_i = \frac{\alpha_i\beta_i\gamma_i\ldots\nu_i}{(\alpha_i-\alpha_1)(\alpha_i-\alpha_2)\ldots(\alpha_i-\alpha_n)}.$$

These are of the same form as the expressions obtained by Jacobi and given on page 202 of the "*Vorlesungen über Dynamik.*" Let σ and τ denote any two of the constants $\alpha, \beta, \gamma \ldots \nu$; then we have, by subtracting the equation whose coefficients are $\frac{1}{\sigma}$ from that having the coefficients $\frac{1}{\tau}$,

$$\frac{x_1(\sigma_1-\tau_1)}{\tau_1\sigma_1} + \frac{x_2(\sigma_2-\tau_2)}{\tau_2\sigma_2} + \ldots \frac{x_n(\sigma_n-\tau_n)}{\tau_n\sigma_n} = 0,$$

or, since $\tau_1-\sigma_1 = \sigma_2-\tau_2 = \&c.$,

$$\frac{x_1}{\tau_1\sigma_1} + \frac{x_2}{\tau_2\sigma_2} + \ldots \frac{x_n}{\tau_n\sigma_n} = 0,$$

which is the same as equation (4) of the "*Vorlesungen.*" We have now the series of equations

47. $$\begin{aligned} \frac{x_1}{\alpha_1\beta_1} + \frac{x_2}{\alpha_2\beta_2} + \ldots \frac{x_n}{\alpha_n\beta_n} &= 0, \\ \frac{x_1}{\alpha_1\gamma_1} + \frac{x_2}{\alpha_2\gamma_2} + \ldots \frac{x_n}{\alpha_n\gamma_n} &= 0, \\ \cdots\cdots\cdots\cdots \\ \frac{x_1}{\alpha_1\nu_1} + \frac{x_2}{\alpha_2\nu_2} + \ldots \frac{x_n}{\alpha_n\nu_n} &= 0. \end{aligned}$$

Define the quantity p_τ by the equation

$$p_\tau = \frac{x_1}{\tau_1^2} + \frac{x_2}{\tau_2^2} + \frac{x_3}{\tau_3^2} + \ldots \frac{x_n}{\tau_n^2},$$

and in particular

$$p_\alpha = \frac{x_1}{\alpha_1^2} + \frac{x_2}{\alpha_2^2} + \ldots \frac{x_n}{\alpha_n^2}.$$

Multiply this last by $\frac{\partial \nabla}{\partial \alpha_1^{-1}}$, and the equations 47 by $\frac{\partial \nabla}{\partial \beta_1^{-1}}$, $\frac{\partial \nabla}{\partial \gamma_1^{-1}}$, &c., respectively, and add the products; we have then

$$\frac{\partial \nabla}{\partial \alpha_1^{-1}} p_\alpha = \frac{x_1}{\alpha_1}\left[\frac{1}{\alpha_1}\frac{\partial \nabla}{\partial \alpha_1^{-1}} + \frac{1}{\beta_1}\frac{\partial \nabla}{\partial \beta_1^{-1}} + \ldots \frac{1}{\nu_1}\frac{\partial \nabla}{\partial \nu_1^{-1}}\right]$$

all the other terms vanishing by virtue of the well-known properties of determinants, that is

$$p_\alpha \frac{\partial \nabla}{\partial \alpha_1^{-1}} = \frac{x_1}{\alpha_1}\nabla,$$

and, similarly, we obtain

$$p_\alpha \frac{\partial \nabla}{\partial \alpha^{-1}} = \frac{x_2}{\alpha_2}\nabla,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$p_\alpha \frac{\partial \nabla}{\partial \alpha_n^{-1}} = \frac{x_n}{\alpha_n}\nabla.$$

Adding these, we have, since

$$\frac{x_1}{\alpha_1} + \frac{x_2}{\alpha_2} + \ldots \frac{x_n}{\alpha_n} = 1,$$

$$p_\alpha = \frac{\nabla}{\frac{\partial \nabla}{\partial \alpha_1^{-1}} + \frac{\partial \nabla}{\partial \alpha_2^{-1}} + \ldots \frac{\partial \nabla}{\partial \alpha_n^{-1}}},$$

Now, by merely changing rows into columns, and conversely, in the determinant ∇, we readily see that we must have, by virtue of equation 45,

$$\nabla = \frac{(\alpha_1 - \beta_1)(\alpha_1 - \gamma_1)\ldots(\alpha_1 - \nu_1)}{\alpha_1\alpha_2\alpha_3\ldots\alpha_n}\left[\frac{\partial \nabla}{\partial \alpha_1^{-1}} + \frac{\partial \nabla}{\partial \alpha_2^{-1}} + \ldots \frac{\partial \nabla}{\partial \alpha_n^{-1}}\right];$$

this reduces the above value of p_α to

48. $$p_\alpha = \frac{(\alpha_1 - \beta_1)(\alpha_1 - \gamma_1)\ldots(\alpha_1 - \nu_1)}{\alpha_1\alpha_2\alpha_3\ldots\alpha_n},$$

and gives, for the general value p_τ,

49. $$p_\tau = \frac{(\tau_1 - \alpha_1)(\tau_1 - \beta_1)\ldots(\tau_1 - \nu_1)}{\tau_1\tau_2\tau_3\ldots\tau_n}.$$

All of the other relations existing between the quantities x_1, x_2, &c., and $\alpha, \beta, \ldots \nu$ can be readily obtained, but it would be out of place to continue

the investigation any further in this paper. The values of x^2, y^2 and z^2 obtained from equation 41, by application of the formulæ of 46, are now

$$x^2 = \frac{(a^2+\lambda_1)(a^2+\lambda_2)(a^2+\lambda_3)}{(a^2-b^2)(a^2-c^2)},$$
$$y^2 = \frac{(b^2+\lambda_1)(b^2+\lambda_2)(b^2+\lambda_3)}{(b^2-c^2)(b^2-a^2)},$$
$$z^2 = \frac{(c^2+\lambda_1)(c^2+\lambda_2)(c^2+\lambda_3)}{(c^2-a^2)(c^2-b^2)};$$

and also for E, F and G we have

$$E^2 = \frac{1}{4}\cdot\frac{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)},$$
$$F^2 = \frac{1}{4}\cdot\frac{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)}{(a^2+\lambda_2)(b^2+\lambda_2)(c^2+\lambda_2)},$$
$$G^2 = \frac{1}{4}\cdot\frac{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}{(a^2+\lambda_3)(b^2+\lambda_3)(c^2+\lambda_3)},$$

The equation of continuity $\nabla^2\phi = 0$ also takes the well-known form

$$(\lambda_2-\lambda_3)\frac{\partial^2\varphi}{\partial\omega_1^2} + (\lambda_3-\lambda_1)\frac{\partial^2\varphi}{\partial\omega_2^2} + (\lambda_1-\lambda_2)\frac{\partial^2\varphi}{\partial\omega_3^2} = 0,$$

where

$$\omega_1 = \int\frac{\partial\lambda_1}{\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}},$$

ω_2 and ω_3 containing λ_2 and λ_3, respectively, instead of λ_1. We are now in the position to examine the case of the motion of an ellipsoid in the fluid, and the resulting motion of the fluid particles, subjects which will be treated fully in a subsequent paper. The transformed equations of motion, as I have given them here, differ in form from those given by Clebsch, and have been obtained by a slightly different process, but will of course lead to the same results as do those of Clebsch. The investigation of the form of ϕ for this case is given in a very elegant manner by Kirchhoff in his Physik.

BALTIMORE, *March 28th, 1879.*